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Gauge Freedom in the N-body problem of Celestial Mechanics.

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Abstract

We summarise research reported in (Efroimsky 2002, 2003; Efroimsky & Goldreich 2003a,b) and develop its application to planetary equations in non-inertial frames.

Whenever a standard system of six planetary equations (in the Lagrange, Delaunay, or other form) is employed, the trajectory resides on a 9(N-1)-dimensional submanifold of the 12(N-1)-dimensional space spanned by the orbital elements and their time derivatives. The freedom in choosing this submanifold reveals an internal symmetry inherent in the description of the trajectory by orbital elements. This freedom is analogous to the gauge invariance of electrodynamics. In traditional derivations of the planetary equations this freedom is removed by hand through the introduction of the Lagrange constraint, either explicitly (in the variation-of-parameters method) or implicitly (in the Hamilton-Jacobi approach). This constraint imposes the condition that the orbital elements osculate the trajectory, i.e., that both the instantaneous position and velocity be fit by a Keplerian ellipse (or hyperbola). Imposition of any supplementary constraint different from that of Lagrange (but compatible with the equations of motion) would alter the mathematical form of the planetary equations without affecting the physical trajectory.

For coordinate-dependent perturbations, any gauge different from that of Lagrange makes the Delaunay system non-canonical. In a more general case of disturbances dependent also upon velocities, there exists a "generalised Lagrange gauge" wherein the Delaunay system is symplectic (and the orbital elements are osculating in the phase space). This gauge reduces to the regular Lagrange gauge for perturbations that are velocity-independent.

We provide a practical example illustrating how the gauge formalism considerably simplifies the calculation of satellite motion about an oblate precessing planet.

1 Introduction

1.1 Prefatory notes

On the 6-th of November 1766 young geometer Joseph-Louis Lagrange, [1] invited from Turin at d'Alembert's recommendation by King Friedrich the Second, succeeded Euler as the Director of Mathematics at the Berlin Academy. Lagrange held the position for 20 years, and this fruitful period of his life was marked by an avalanche of excellent results, and by three honourable prizes received by him from the Académie des Sciences of Paris. All three prizes (one of which he shared with Euler) were awarded to Lagrange for his contributions to celestial mechanics. Among these contributions was a method initially developed by Lagrange for his studies of planet-perturbed cometary orbits and only later applied to planetary motion (Lagrange 1778, 1783, 1788, 1808, 1809, 1810). The method is based on an elegant mathematical novelty, the variation of parameters emerging in solutions of differential equations. The first known instances of this tool being employed are presented in a paper on Jupiter's and Saturn's mutual disturbances, submitted by Euler to a competition held by the French Academy of Sciences (Euler 1748), and in the treatise on the Lunar motion, published by Euler in 1753 in St.Petersburg (Euler 1753). However it was Lagrange who revealed the full power of the method.

Below we shall demonstrate that the equations for the instantaneous orbital elements possess a hidden symmetry similar to the gauge symmetry of electrodynamics. Derivation of the Lagrange system involves a mathematical operation equivalent to the choice of a specific gauge. As a result, trajectories get constrained to some 9-dimensional submanifold in the 12-dimensional space constituted by the orbital elements and their time derivatives. However, the choice of this submanifold is essentially ambiguous, and this ambiguity gives birth to an internal symmetry. The symmetry is absent in the 2-body case, but comes into being in the N-body setting ($N \geq 3$) where each body follows an ellipse of varying shape whose time evolution contains an inherent ambiguity.

For a simple illustration of this point imagine two coplanar ellipses sharing one focus. Suppose they rotate at different rates in their common plane. Let a planet be located at one of the intersection points of these ellipses. The values of the elliptic elements needed to describe its trajectory would depend upon which ellipse was chosen to parameterise the orbit. Either set would be equally legitimate and would faithfully describe the physical trajectory. Thus we see that there exists an infinite number of ways of dividing the actual planet's movement into motion along its orbit and the simultaneous evolution of the orbit. Although the physical trajectory is unique, its description (parametrisation in terms of Kepler's elements) is not. A map between two different (though physically-equivalent) sets of orbital elements is a symmetry transformation (a gauge transformation, in physicists' jargon).

Lagrange never dwelled on that point. However, in his treatment (based on direct application of the method of variation of parameters) he passingly introduced a convenient mathematical condition which removed the said ambiguity. This condition and possible alternatives to it will be the topics of Sections 1 - 3 of this paper.

[1] The real name of the young man invited in 1766 to the Prussian court was Giuseppe Lodovico Lagrangia. It was only several years down the road that he became Joseph-Louis Lagrange.

In 1834 - 1835 Hamilton put forward his theory of canonical transformations. Several years later this approach was furthered by Jacobi who brought Hamilton's technique into astronomy and, thereby, worked out a new method of deriving the planetary equations (Jacobi 1866), a method that was soon accepted as standard. Though the mathematical content of the Hamilton-Jacobi theory is impeccably correct, its application to astronomy contains a long overlooked aspect that needs attention. This aspect is: where is the Lagrange constraint tacitly imposed, and what happens if we impose a different constraint? This issue will be addressed in Section 4 of our paper.

1.2 Osculating Elements vs Orbital Elements

We start in the spirit of Lagrange. Before addressing the N-particle case, Lagrange referred to the reduced 2-body problem,

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\frac{\vec{\mathbf{r}}}{r} = 0 \quad ,$$
$$\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_{planet} - \vec{\boldsymbol{r}}_{sun} \quad , \qquad \mu \equiv G(m_{planet} + m_{sun}) \quad , \tag{1}$$

whose generic solution, a Keplerian ellipse or a hyperbola, can be expressed, in some fixed Cartesian frame, as

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{f}}(C_1, ..., C_6,\, t) \quad , \qquad \dot{\vec{\boldsymbol{r}}} = \vec{\mathbf{g}}(C_1, ..., C_6,\, t) \quad , \tag{2}$$

where

$$\vec{\mathbf{g}} \equiv \left(\frac{\partial \vec{\boldsymbol{f}}}{\partial t}\right)_{C=const} . \tag{3}$$

Since the problem (1) is constituted by three second-order differential equations, its general solution naturally contains six adjustable constants C_i. At this point it is irrelevant which particular set of the adjustable parameters is employed. (It may be, for example, a set of Lagrange or Delaunay orbital elements or, alternatively, a set of initial values of the coordinates and velocities.)

Following Lagrange (1808, 1809, 1810), we employ $\vec{\boldsymbol{f}}$ as an ansatz for a solution of the N-particle problem, the disturbing force acting at a particle being denoted by $\Delta\vec{\mathbf{F}}$:[2]

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^2}\frac{\vec{\mathbf{r}}}{r} = \Delta\vec{\mathbf{F}} \quad . \tag{4}$$

Now the "constants" become time dependent:

$$\vec{\mathbf{r}} = \vec{\boldsymbol{f}}(C_1(t), ..., C_6(t),\, t) \quad , \tag{5}$$

whence the velocity

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \frac{\partial \vec{\boldsymbol{f}}}{\partial t} + \sum_i \frac{\partial \vec{\boldsymbol{f}}}{\partial C_i}\frac{dC_i}{dt} = \vec{\mathbf{g}} + \sum_i \frac{\partial \vec{\boldsymbol{f}}}{\partial C_i}\frac{dC_i}{dt} \quad , \tag{6}$$

[2]Our treatment covers disturbing forces $\Delta\vec{\mathbf{F}}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t)$ that are arbitrary vector-valued functions of position, velocity, and time.

acquires a new input, $\sum(\partial\vec{\boldsymbol{f}}/\partial C_i)(dC_i/dt)$.

Substitution of (5) and (6) into the perturbed equation of motion (4) leads to three independent scalar second-order differential equations which contain one independent parameter, time, and six functions $C_i(t)$. These are to be found from the said three equations, and this makes the essence of the variation-of-parameters (VOP) method. However, the latter task cannot be accomplished in a unique way because the number of variables exceeds, by three, the number of equations. Thence, though the *physical* trajectory (comprised by the locus of points in the Cartesian frame and by the values of velocity at each of these points) is unique, its parametrisation through the orbital elements is ambiguous. This circumstance was appreciated by Lagrange who amended the system, by hand, with three independent conditions,

$$\sum_i \frac{\partial\vec{\boldsymbol{f}}}{\partial C_i}\,\frac{dC_i}{dt}\;=\;0 \quad , \tag{7}$$

and thus made it solvable. His choice of constraints was motivated by both physical considerations and mathematical expediency. Since, physically, the set of functions $(C_1(t), ..., C_6(t))$ can be interpreted as parameters of an instantaneous ellipse, in a bound-orbit case, or an instantaneous hyperbola, in a fly-by situation, Lagrange found it natural to make the instantaneous orbital elements C_i osculating. His constraint (7) fixes the instantaneous ellipse (or hyperbola) defined by $(C_1(t), ..., C_6(t))$ such that, at each moment of time, it coincides with the unperturbed (two-body) orbit that the body would follow if the disturbances were to cease instantaneously. This way, Lagrange restricted his use of the orbital elements to elements that osculate in the reference frame wherein ansatz (5) is employed. Lagrange never explored alternative options; he simply imposed (7) and used it to derive his famous system of equations for orbital elements.

Such a restriction, though physically motivated, is completely arbitrary from the mathematical point of view. While the imposition of (7) considerably simplifies the subsequent calculations it in no way influences the shape of the physical trajectory and the rate of motion along it. A choice of any other supplementary constraint

$$\sum_i \frac{\partial\vec{\boldsymbol{f}}}{\partial C_i}\,\frac{dC_i}{dt}\;=\;\vec{\boldsymbol{\Phi}}(C_{1,...,6}\,,\,t) \quad , \tag{8}$$

$\vec{\boldsymbol{\Phi}}$ being an arbitrary function of time and parameters C_i, would lead to the same physical orbit and the same velocities.[3] Substitution of the Lagrange constraint (7) by its generalisation (8) would not influence the motion of the body but would alter its mathematical description (i.e., would entail different solutions for the orbital elements). Such invariance of a physical picture under a change of parametrisation is called gauge freedom or gauge symmetry. It parallels a similar phenomenon well known in electrodynamics and, therefore, has similar consequences. On the one hand, a good choice of gauge often simplifies solution of the equations of motion. (In Section 3 we provide a specific application to motion in an accelerated coordinate system.) On the other hand, a so-called "gauge shift" will occur in the course of numerical orbit computation (Murison & Efroimsky 2003).

Derivation of the conventional Lagrange and Delaunay planetary equations by the VOP method incorporates the Lagrange constraint (Brouwer & Clemence 1961). Both systems

[3]In principle, one may endow $\vec{\boldsymbol{\Phi}}$ also with dependence upon the parameters' time derivatives of all orders. This would yield higher-than-first-order time derivatives of the C_i in subsequent developments requiring additional initial conditions, beyond those on $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$, to be specified in order to close the system. We avoid this unnecessary complication by restricting $\vec{\boldsymbol{\Phi}}$ to be a function of time and the C_i .

of equations get altered under a different gauge choice as we now show. The essence of a derivation suitable for a general choice of gauge starts with (6) from which the formula for the acceleration follows:

$$\frac{d^2\vec{\boldsymbol{r}}}{dt^2} \;=\; \frac{\partial\vec{\mathbf{g}}}{\partial t} \;+\; \sum_i \frac{\partial\vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \;=\; \frac{\partial^2\vec{\boldsymbol{f}}}{\partial^2 t} \;+\; \sum_i \frac{\partial\vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \quad . \tag{9}$$

Together with the equation of motion (4), it yields:

$$\frac{\partial^2\vec{\boldsymbol{f}}}{\partial t^2} \;+\; \frac{\mu}{r^2}\,\frac{\vec{\boldsymbol{f}}}{r} \;+\; \sum_i \frac{\partial\vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \;=\; \Delta\vec{\mathbf{F}} \quad , \qquad r \;\equiv\; |\vec{\boldsymbol{r}}| \;=\; |\vec{\boldsymbol{f}}| \quad . \tag{10}$$

The vector function $\vec{\boldsymbol{f}}$ was from the beginning introduced as a Keplerian solution to the two-body problem; hence it must obey the unperturbed equation (1) . On these grounds the above formula simplifies to:

$$\sum_i \frac{\partial\vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;=\; \Delta\vec{\mathbf{F}} \;-\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \quad . \tag{11}$$

This equation describes the perturbed motion in terms of the orbital elements. Together with constraint (8) it constitutes a well-defined system which may be solved with respect to dC_i/dt . An easy way to do this is to use the elegant trick suggested by Lagrange: to multiply the equation of motion by $\partial\vec{\boldsymbol{f}}/\partial C_n$ and to multiply the constraint by $-\partial\vec{\mathbf{g}}/\partial C_n$. The former operation results in

$$\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\left(\sum_j \frac{\partial\vec{\mathbf{g}}}{\partial C_j}\,\frac{dC_j}{dt}\right) \;=\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\;\Delta\vec{\mathbf{F}} \;-\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{d\vec{\boldsymbol{\Phi}}}{dt}\;, \tag{12}$$

while the latter gives

$$-\;\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\left(\sum_j \frac{\partial\vec{\boldsymbol{f}}}{\partial C_j}\,\frac{dC_j}{dt}\right) \;=\; -\;\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\;\vec{\boldsymbol{\Phi}} \quad . \tag{13}$$

Having summed these two equalities, we arrive to:

$$\sum_j \,[C_n\; C_j]\;\frac{dC_j}{dt} \;=\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\Delta\vec{\mathbf{F}} \;-\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{d\vec{\boldsymbol{\Phi}}}{dt} \;-\; \frac{\partial\vec{\mathbf{g}}}{\partial C_n}\;\vec{\boldsymbol{\Phi}} \quad , \tag{14}$$

$[C_n\; C_j]$ standing for the unperturbed (i.e., defined as in the two-body case) Lagrange brackets:

$$[C_n\; C_j] \;\equiv\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial\vec{\mathbf{g}}}{\partial C_j} \;-\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_j}\,\frac{\partial\vec{\mathbf{g}}}{\partial C_n} \quad . \tag{15}$$

It was agreed above that $\vec{\boldsymbol{\Phi}}$ is a function of time and of the "constants" C_i , but not of their time derivatives. Under this convention, the above equation may be shaped into a more convenient form:

$$\sum_j \left(\,[C_n\; C_j] \;+\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial\vec{\boldsymbol{\Phi}}}{\partial C_j}\,\right)\frac{dC_j}{dt} \;=\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\Delta\vec{\mathbf{F}} \;-\; \frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial\vec{\boldsymbol{\Phi}}}{\partial t} \;-\; \frac{\partial\vec{\mathbf{g}}}{\partial C_n}\;\vec{\boldsymbol{\Phi}} \quad . \tag{16}$$

This is the most general form of the gauge-invariant perturbation equations of celestial mechanics. In the Lagrange gauge, when the $\vec{\boldsymbol{\Phi}}$ terms are absent, we can obtain an immediate solution for the individual dC_i/dt by exploiting the well known expression for the Poisson-bracket matrix which is inverse to the Lagrange-bracket one and is offered in the literature for the two-body problem. (Be mindful that our brackets (15) are defined in the same manner as in the two-body case; they contain only functions $\vec{\boldsymbol{f}}$ and $\vec{\mathbf{g}}$ that are defined in the unperturbed, two-body, setting.) In an arbitrary gauge the presence of the term proportional to $\partial\vec{\boldsymbol{\Phi}}/\partial C_j$ on the left-hand side of (16) complicates the solution for dC_i/dt, but only to the extent of requiring the resolution of a set of six simultaneous linear algebraic equations.

To draw to a close, we again emphasise that, for fixed interactions and initial conditions, all possible (i.e., compatible and sufficient) choices of gauge conditions expressed by the vector function $\vec{\boldsymbol{\Phi}}$ lead to a physically equivalent picture. In other words, the real trajectory is invariant under reparametrisations permitted by the ambiguity of the choice of gauge. This invariance has the following meaning. Suppose the equations of motion for $C_{1,...,6}$, with some gauge condition $\vec{\boldsymbol{\Phi}}$ imposed, render the solution $C_{1,...,6}(t)$. The same equations of motion, with another gauge $\tilde{\vec{\boldsymbol{\Phi}}}$ enforced, furnish the solution $\tilde{C}_i(t)$ that has a different functional form. Despite this difference, both solutions, $C_i(t)$ and $\tilde{C}_i(t)$, when substituted back in (5), yield the same curve $\vec{\mathbf{r}}(\mathbf{t})$ with the same velocities $\dot{\vec{\mathbf{r}}}(t)$. In mathematics this situation is called a fibre bundle, and it gives birth to a 1-to-1 map of $C_i(t)$ onto $\tilde{C}_i(t)$, which is merely a reparametrisation. In physics this map is called a gauge transformation. The entire set of these reparametrisations constitute a group of symmetry known as a gauge group.

Just as in electrodynamics, where the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ stay invariant under gradient transformations of the 4-potential A^{μ} , so the invariance of the orbit implements itself through the form-invariance of expression (5) under the afore mentioned map. The vector $\vec{\boldsymbol{r}}$ and its full time derivative $\dot{\vec{\boldsymbol{r}}}$, play the role of the physical fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, while the Keplerian coordinates $C_1, ..., C_6$ play the role of the 4-potential A^{μ} .

A comprehensive discussion of all the above-raised issues can be found in Efroimsky (2002, 2003). Interconnection between the internal symmetry and multiple time scales in celestial mechanics is touched upon in Newman & Efroimsky (2003).

2 Planetary equations in an arbitrary gauge

2.1 Lagrangian and Hamiltonian Perturbations

We can proceed further by restricting the class of perturbations we consider to those in which $\Delta\vec{\mathbf{F}}$ is derivable from a perturbed Lagrangian. This restricted class is still broad enough to encompass most applications of celestial-mechanics perturbation theory.

Let the unperturbed dynamics be described by Lagrangian $L_o(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) = \dot{\vec{\mathbf{r}}}^{\,2}/2 - U(\vec{\mathbf{r}}\,,\, t)$, canonical momentum $\vec{\mathbf{p}} = \dot{\vec{\mathbf{r}}}$ and Hamiltonian $H_o(\vec{\mathbf{r}}, \vec{\mathbf{p}}, t) = \vec{\mathbf{p}}^{\,2}/2 + U(\vec{\mathbf{r}}\,,\, t)$. Disturbed motion will be described by the new, perturbed, functions:

$$L(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) \;=\; L_o \;+\; \Delta L \;=\; \frac{\dot{\vec{\mathbf{r}}}^{\,2}}{2} \;-\; U(\vec{\mathbf{r}},\; t) \;+\; \Delta L(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) \quad , \tag{17}$$

$$\vec{\mathbf{p}} \;=\; \dot{\vec{\mathbf{r}}} \;+\; \frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}} \quad , \tag{18}$$

$$H(\vec{\mathbf{r}},\,\vec{\mathbf{p}},\,t)\;=\;\vec{\mathbf{p}}\,\dot{\vec{\mathbf{r}}}\,-\,L\;=\;\frac{\vec{\mathbf{p}}^{\;2}}{2}\,+\,U\,+\,\Delta H\;\;,\qquad \Delta H(\vec{\mathbf{r}},\,\vec{\mathbf{p}},\,t)\;\equiv\;-\,\Delta L\,-\,\frac{1}{2}\left(\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)^2\;\;,\tag{19}$$

ΔH being introduced as a variation of the functional form, i.e., as $\Delta H\;\equiv\;H(\vec{\mathbf{r}},\,\vec{\mathbf{p}},\,t)\,-\,H_o(\vec{\mathbf{r}},\,\vec{\mathbf{p}},\,t)$. The Euler-Lagrange equations written for the perturbed Lagrangian (17) will give:

$$\ddot{\vec{\mathbf{r}}}\;=\;-\,\frac{\partial U}{\partial\vec{\mathbf{r}}}\,+\,\Delta\vec{\mathbf{F}}\quad,\tag{20}$$

where the new term

$$\Delta\vec{\mathbf{F}}\;\equiv\;\frac{\partial\,\Delta L}{\partial\vec{\mathbf{r}}}\,-\,\frac{d}{dt}\left(\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)\tag{21}$$

is the disturbing force. Expression (21) reveals that whenever the Lagrangian perturbation is velocity independent, it plays the role of the disturbing function. Generally, though, the disturbing force is not equal to the gradient of ΔL but has an extra term generated by the velocity dependence.

Examples in which a velocity dependent ΔL has been used in a celestial mechanics setting include: the treatment of inertial forces in a coordinate system tied to the spin axis of a precessing planet (Goldreich 1965); the velocity dependent corrections to Newton's law of gravity in the relativistic two-body problem (Brumberg 1992).

2.2 Gauge-invariant planetary equations

When the expression (21) for the most general force emerging within the Lagrangian formalism is substituted into the gauge-invariant perturbation equation (14), it yields:

$$\sum_j\,[C_n\;C_j]\,\frac{dC_j}{dt}\;=\;\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial\,\Delta L}{\partial\vec{\boldsymbol{r}}}\,-\,\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{d}{dt}\left(\vec{\boldsymbol{\Phi}}\,+\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)\,-\,\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\,\vec{\boldsymbol{\Phi}}\quad.\tag{22}$$

Since, for a velocity-dependent disturbance, the chain rule

$$\frac{\partial\,\Delta L}{\partial C_n}\;=\;\frac{\partial\,\Delta L}{\partial\vec{\boldsymbol{r}}}\,\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,+\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\boldsymbol{r}}}}\,\frac{\partial\dot{\vec{\mathbf{r}}}}{\partial C_n}\;=\;\frac{\partial\,\Delta L}{\partial\vec{\boldsymbol{r}}}\,\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,+\,\frac{\partial\Delta L}{\partial\dot{\vec{\boldsymbol{r}}}}\,\frac{\partial(\vec{\mathbf{g}}\,+\,\vec{\boldsymbol{\Phi}})}{\partial C_n}\quad,\tag{23}$$

takes place, formula (22) may be re-shaped into

$$\sum_j\,[C_n\;C_j]\,\frac{dC_j}{dt}\;=\;\frac{\partial\,\Delta L}{\partial C_n}\,-\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\boldsymbol{r}}}}\,\frac{\partial\vec{\boldsymbol{\Phi}}}{\partial C_n}\,-\,\left(\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{d}{dt}\,-\,\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right)\left(\vec{\boldsymbol{\Phi}}\,+\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)\quad.\tag{24}$$

Next we group terms so that the gauge function $\vec{\boldsymbol{\Phi}}$ everywhere appears added to $\partial(\Delta L)/\partial\dot{\vec{\mathbf{r}}}$, and bring the term proportional to dC_j/dt to the left hand side of the equation:

$$\sum_j\,\left([C_n\;C_j]\,+\,\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial}{\partial C_j}\left(\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\,+\,\vec{\boldsymbol{\Phi}}\right)\right)\frac{dC_j}{dt}\;=$$

$$\frac{\partial}{\partial C_n}\left[\Delta L\,+\,\frac{1}{2}\left(\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)^2\right]\,-\,\left(\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\,+\,\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\,\frac{\partial}{\partial t}\,+\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\,\frac{\partial}{\partial C_n}\right)\left(\vec{\boldsymbol{\Phi}}\,+\,\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right)\quad.\tag{25}$$

These modifications help us to recognise the special nature of the gauge $\vec{\Phi} = -\partial(\Delta L)/\partial\dot{\vec{r}}$ which will be the subject of discussion in the next subsection.

Contrast (25) with (16): while (16) expresses the VOP method in the most general form it can have in terms of the disturbing force $\Delta\vec{F}(\vec{r}, \dot{\vec{r}}, t)$, equation (25) renders the most general form in the language of a Lagrangian perturbation $\Delta L(\vec{r}, \dot{\vec{r}}, t)$.

The applicability of so generalised planetary equations in analytical calculations is complicated by the nontrivial nature of the left-hand sides of (16) and (25). Nevertheless, the structure of these left-hand sides leaves room for analytical simplification in particular situations. One such situation is when the gauge is chosen to be

$$\vec{\Phi} = -\frac{\partial \Delta L}{\partial \dot{\vec{r}}} + \vec{\eta}(\vec{r}, t) \quad , \tag{26}$$

$\vec{\eta}(\vec{r}, t)$ being an arbitrary vector function linear in $\vec{r}$. (It may be, for example, proportional to $\vec{r}$ or may be equal, say, to a cross product of $\vec{r}$ by some time-dependent vector.) Under these circumstances the left-hand side in (25) reduces to the Lagrange brackets. The situation becomes especially simple when $\partial\Delta L/\partial\dot{\vec{r}}$ happens to be linear in $\vec{r}$, in which case we may put $\vec{\eta}(\vec{r}, t) = \partial\Delta L/\partial\dot{\vec{r}}$ and, thus, employ the trivial Lagrange gauge $\vec{\Phi} = 0$ instead of the generalised Lagrange gauge. We shall encounter one such example in section 3.4.

As already stressed above, the Lagrange brackets are gauge-invariant because functions $\vec{f}$ and $\vec{g}$ used in are defined within the unperturbed, two-body, problem (1 - 3) that lacks gauge freedom. For this reason, one may exploit, to solve (25), the well-known expression for the inverse of matrix $[C_i \; C_j]$. Its elements are simplest (and are either zero or unity) when one chooses as the "constants" the Delaunay set of orbital variables (Plummer 1918):

$$C_i = \{ L , G , H , M_o , \omega , \Omega \}$$
$$L \equiv \sqrt{\mu\, a} \quad , \quad G \equiv \sqrt{\mu\, a\, (1 - e^2)} \quad , \quad H \equiv \sqrt{\mu\, a\, (1 - e^2)}\, \cos i \quad , \tag{27}$$

where $\mu \equiv G(m_{sun} + m_{planet})$ and $(e , a , M_o , \omega , \Omega , i)$ are the Keplerian elements: e and a are the eccentricity and major semiaxis, M_o is the mean anomaly at epoch, and the Euler angles ω , Ω , i are the the argument of pericentre, the longitude of the ascending node, and the inclination, respectively.

The simple forms of the Lagrange and Poisson brackets in Delaunay elements is the proof of these elements' canonicity in the unperturbed, two-body, problem: the Delaunay elements give birth to three canonical pairs (Q_i, P_i) corresponding to a vanishing Hamiltonian: $(L , -M_o)$, $(G , -\omega)$, $(H , -\Omega)$. In a perturbed setting, when only a position-dependent disturbing function $R(\vec{r}, t)$ is "turned on", it can be expressed through the Lagrangian and Hamiltonian perturbations in a simple manner, $R(\vec{r}, t) = \Delta L(\vec{r}, t) = -\Delta H(\vec{r}, t)$, as can be seen from the formulae presented in the previous subsection. Under these circumstances, the Delaunay elements remain canonical, provided the Lagrange gauge is imposed (Brouwer & Clemence 1961). This long known fact can also be derived from our equation (25): if we put $\vec{\Phi} = 0$ and assume ΔL velocity-independent, we arrive to

$$\sum_j [C_n \; C_j] \frac{dC_j}{dt} = \frac{\partial \Delta L}{\partial C_n} \quad , \tag{28}$$

where

$$\Delta L = \Delta L\left(\vec{f}(C , t) , t\right) = R\left(\vec{f}(C , t) , t\right) = -\Delta H\left(\vec{f}(C , t) , t\right) \quad . \tag{29}$$

This, in its turn, results in the well known Lagrange system of planetary equations, provided the parameters C_i are chosen as the Kepler elements. In case they are chosen as the Delaunay elements, then (28) leads to the standard Delaunay equations, i.e., to a symplectic system wherein the pairs $(L\,,\,-M_o)\,,\;(G\,,\,-\omega)\,,\;(H\,,\,-\Omega)$ again play the role of canonical variables, but the Hamiltonian, in distinction to the unperturbed case, no longer vanishes, instead being equal to $\Delta H \;=\; -\,\Delta L$.

In our more general case, the perturbation depends also upon velocities (and, therefore, ΔL is no longer equal to $-\,\Delta H$). Beside this, the gauge $\vec{\mathbf{\Phi}}$ is set arbitrary. As demonstrated in Efroimsky (2002, 2003), under these circumstances the gauge-invariant Delaunay-type system is no longer symplectic. However, it turns out that this system regains the canonical form in one special gauge, one that coincides with the Lagrange gauge when the perturbation bears no velocity dependence. The issue is explained at length in our previous papers (Efroimsky & Goldreich 2003a,b). Here we offer a brief synopsis of this study.

2.3 Generalised Lagrange gauge wherein the Delaunay-type system is canonical

Equation (22) was cast in the shape of (25) not only to demonstrate the special nature of the gauge

$$\vec{\mathbf{\Phi}} \;=\; -\,\frac{\partial\,\Delta L}{\partial \dot{\vec{\mathbf{r}}}} \quad , \tag{30}$$

but also to single out the terms in the square brackets on the right-hand side of (25): together, these terms give exactly the Hamiltonian perturbation. Thus we come to the conclusion that in the special gauge (30) our equation (25) simplifies to

$$\sum_j\,[C_n\;C_j]\;\frac{dC_j}{dt}\;=\;-\;\frac{\partial\;\Delta H}{\partial C_n}\quad . \tag{31}$$

As emphasised in the preceding subsection, the gauge invariance of definition (15) enables us to use the standard (Lagrange-gauge) expressions for $[C_n\,C_j]^{-1}$ to get the planetary equations from (31).

Comparing (31) with (28), we see that in the general case of an arbitrary $\Delta L(\vec{\mathbf{r}},\,\dot{\vec{\mathbf{r}}},\,t)$ one arrives from (31) to the same equations as from (28), except that now they contain $-\,\Delta H$ instead of ΔL. When the orbit is parametrised by the Delaunay variables, those equations take the form:

$$\frac{dL}{dt}=\frac{\partial\,\Delta H}{\partial(\,-\,M_o)}\;\;,\qquad\frac{d(\,-\,M_o)}{dt}\;=\;-\;\frac{\partial\,\Delta H}{\partial L}\quad , \tag{32}$$

$$\frac{dG}{dt}\;=\;\frac{\partial\,\Delta H}{\partial(\,-\,\omega)}\quad,\qquad\frac{d(\,-\,\omega)}{dt}\;=\;-\;\frac{\partial\,\Delta H}{\partial G}\quad , \tag{33}$$

$$\frac{dH}{dt}\;=\;\frac{\partial\,\Delta H}{\partial(\,-\,\Omega)}\quad,\qquad\frac{d(\,-\,\Omega)}{dt}\;=\;-\;\frac{\partial\,\Delta H}{\partial H}\quad . \tag{34}$$

which is a symplectic system.[4] For this reason we name this special gauge the "generalised Lagrange gauge". In any different gauge $\vec{\mathbf{\Phi}}$ the equations for the Delaunay variables

[4] In this system H stands not for the Hamiltonian but for one of the Delaunay elements.

would contain $\vec{\Phi}$-dependent terms and would not be symplectic. (Those gauge-invariant equations, for both Lagrange and Delaunay elements are presented in Efroimsky (2002, 2003) and Efroimsky & Goldreich (2003a,b).). This analysis proves the following <u>**THEOREM**</u>: **Though the gauge-invariant equations for Delaunay elements are, generally, not canonical, they become canonical in the "generalised Lagrange gauge".** That this Theorem is not merely a mathematical coincidence but has deep reasons beneath it will be shown in Section 4 where the subject is approached from the Hamilton-Jacobi viewpoint.

The above Theorem gives one example of the gauge formalism being of use: an appropriate choice of gauge can considerably simplify the planetary equations (in this particular case, it makes them canonical).

According to (18), the momentum can be written as

$$\vec{\mathbf{p}} \;=\; \dot{\vec{\mathbf{r}}} \,+\, \frac{\partial \Delta L}{\partial \dot{\vec{\mathbf{r}}}} \;=\; \vec{\mathbf{g}} \,+\, \vec{\mathbf{\Phi}} \,+\, \frac{\partial \Delta L}{\partial \dot{\vec{\mathbf{r}}}} \quad , \tag{35}$$

which, in the generalised Lagrange gauge (30), simply reduces to

$$\vec{\mathbf{p}} \;=\; \vec{\mathbf{g}} \quad . \tag{36}$$

Vector $\vec{\mathbf{g}}$ was introduced back in (2 - 3) to denote the functional dependence of the unperturbed velocity upon the time and the parameters C_i . In the unperturbed, two-body, setting this velocity is equal to the momentum canonically conjugate to the position $\vec{\mathbf{r}}$ (this is obvious from (18), for zero ΔL). This way, in the unperturbed case equality (36) is fulfilled trivially. The fact that it remains valid also under perturbation means that, in the said gauge, the canonical momentum in the disturbed setting is the same function of time and "constants" as in the unperturbed, two-body, case. Thus we have established that the instantaneous Keplerian ellipses (hyperbolae) defined in gauge (30) osculate the trajectory **in phase space**.

Not surprisingly, the generalised Lagrange gauge (30) reduces to $\vec{\mathbf{\Phi}} = 0$ in the simple case of velocity-independent disturbances.

3 Gauge Freedom and Freedom of Frame Choice

3.1 Osculating ellipses described in different frames of reference.

The essence of the VOP method in celestial mechanics is the following. A generic two-body-problem solution expressed by

$$\vec{\boldsymbol{r}} \quad = \quad \vec{\boldsymbol{f}}\,(C,\,t) \quad , \tag{37}$$

$$\left(\frac{\partial \vec{\boldsymbol{f}}}{\partial t}\right)_C \;=\; \vec{\mathbf{g}}\,(C,\,t) \quad , \tag{38}$$

$$\left(\frac{\partial \vec{\mathbf{g}}}{\partial t}\right)_C \;=\; -\,\frac{\mu}{f^2}\,\frac{\vec{\boldsymbol{f}}}{f} \quad , \tag{39}$$

is employed as an ansatz to solve the disturbed problem:

$$\vec{\mathbf{r}} \;=\; \vec{\boldsymbol{f}}(C(t),\,t) \;\; , \tag{40}$$

$$\dot{\vec{\mathbf{r}}} \;=\; \frac{\partial \vec{\boldsymbol{f}}}{\partial t} \;+\; \frac{\partial \vec{\boldsymbol{f}}}{\partial C_i}\,\frac{dC_i}{dt} \;=\; \vec{\mathbf{g}} \;+\; \vec{\boldsymbol{\Phi}} \quad , \tag{41}$$

$$\begin{aligned} \ddot{\vec{\mathbf{r}}} &= \frac{\partial \vec{\mathbf{g}}}{\partial t} \;+\; \frac{\partial \vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \\ &= \;-\;\frac{\mu}{f^2}\,\frac{\vec{\boldsymbol{f}}}{f} \;+\; \frac{\partial \vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \quad . \end{aligned} \tag{42}$$

As evident from (41), our choice of a particular gauge is equivalent to decomposition of the physical motion into a movement with velocity $\vec{\mathbf{g}}$ along the instantaneous ellipse (or hyperbola, in the fly-by case), and a movement associated with the ellipse's (or hyperbola's) deformation that goes at the rate $\vec{\boldsymbol{\Phi}}$. It is then tempting to state that a choice of gauge is equivalent to a choice of an instantaneous comoving reference frame wherein to describe the motion. Such an interpretation is, however, incomplete. Beside the fact that we decouple the physical velocity in a certain proportion between $\vec{\mathbf{g}}$ and $\vec{\boldsymbol{\Phi}}$, it also matters **which** physical velocity (i.e., velocity relative to what frame) is decoupled in this proportion. In other words, our choice of the gauge does not yet exhaust all freedom: we can still choose **in what frame** to write ansatz (40). We may write it in inertial axes or in some accelerated system. For example, in the case of a satellite orbiting an accelerated and precessing planet it is **convenient** to write the ansatz for the planet-related position vector.

The above kinematic formulae (40) - (42) do not yet contain information about our choice of the reference system in which we implement the VOP method. This information shows up at the next stage, when expression (42) is inserted into the dynamical equation of motion $\ddot{\vec{\boldsymbol{r}}} \,=\, -\,(\mu\vec{\boldsymbol{r}}/r^3) \,+\, \Delta\vec{\mathbf{F}}$ to yield

$$\frac{\partial \vec{\mathbf{g}}}{\partial C_i}\,\frac{dC_i}{dt} \;+\; \frac{d\vec{\boldsymbol{\Phi}}}{dt} \;=\; \Delta\vec{\mathbf{F}} \;=\; \frac{\partial\,\Delta L}{\partial\vec{\mathbf{r}}} \;-\; \frac{d}{dt}\left(\frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}}\right) \quad . \tag{43}$$

Complete information about the reference system in which we put the VOP method to work (and, therefore, in which we define the orbital elements C_i) is contained in the expression for the perturbation force $\Delta\vec{\mathbf{F}}$. For example, if the operation is carried out in an inertial coordinate system, $\Delta\vec{\mathbf{F}}$ contains physical forces solely. However, if we wish to implement the VOP approach in a frame moving with a linear acceleration $\vec{\mathbf{a}}$, then $\Delta\vec{\mathbf{F}}$ also contains the inertial force $-\,\vec{\mathbf{a}}$. In case this coordinate system rotates relative to inertial ones at a rate $\vec{\boldsymbol{\mu}}$, then $\Delta\vec{\mathbf{F}}$ also includes the inertial terms $-\,2\,\vec{\boldsymbol{\mu}}\times\dot{\vec{\mathbf{r}}} - \dot{\vec{\boldsymbol{\mu}}}\times\vec{\mathbf{r}} - \vec{\boldsymbol{\mu}}\times(\vec{\boldsymbol{\mu}}\times\vec{\mathbf{r}})$. In considering the motion of a satellite orbiting an oblate precessing planet it is most reasonable, though not obligatory, to apply the method (i.e., to define the time derivative) in axes that precess with the planet. However, this reasonable choice of coordinate system still leaves us with the freedom of gauge nomination.

3.2 Relevant Example

Gauge freedom of the perturbation equations of celestial mechanics finds an immediate practical implementation in the description of test particle motion around an precessing oblate planet (Goldreich, 1965). It is trivial to extend this to account for acceleration of the planet's centre of mass.

Our starting point is the equation of motion in the inertial frame

$$\vec{\mathbf{r}}\,'' \;=\; \frac{\partial U}{\partial \vec{\mathbf{r}}} \quad , \tag{44}$$

where U is the total gravitational potential and time derivatives in the inertial axes are denoted by primes. Suppose that the planet's spin axis precesses at angular rate $\vec{\boldsymbol{\mu}}(t)$ and that the acceleration of its centre of mass is given by $\vec{\mathbf{a}}(t)$. In a coordinate system attached to the planet's centre of mass and precessing with it, inertial forces modify the equation of motion so that it assumes the form:

$$\ddot{\vec{\mathbf{r}}} = \frac{\partial U}{\partial \vec{\mathbf{r}}} - 2\vec{\boldsymbol{\mu}} \times \dot{\vec{\mathbf{r}}} - \dot{\vec{\boldsymbol{\mu}}} \times \vec{\mathbf{r}} - \vec{\boldsymbol{\mu}} \times (\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \;-\; \vec{\mathbf{a}} \;\; , \tag{45}$$

time derivatives in the accelerated frame being denoted by dots.

To implement the VOP approach in terms of the orbital elements defined in the accelerated frame, we note that the disturbing force on the right-hand side of (45) is generated according to (21) by

$$\Delta L\left(\vec{\mathbf{r}},\,\dot{\vec{\mathbf{r}}},\,t\right) \;=\; R \;+\; \dot{\vec{\mathbf{r}}}\cdot(\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \;+\; \frac{1}{2}\,(\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}})\cdot(\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \;-\; \vec{\mathbf{a}} \cdot \vec{\mathbf{r}} \;\; , \tag{46}$$

where we denote by $R(\vec{\mathbf{r}}\,,\,t)$ the gravitational-potential perturbation (which the perturbation of the overall gravitational potential U). Since

$$\frac{\partial\,\Delta L}{\partial \dot{\vec{\mathbf{r}}}} \;=\; \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}} \quad , \tag{47}$$

the corresponding Hamiltonian perturbation reads:

$$\begin{aligned} \Delta H \;&=\; -\;\left[\Delta L \;+\; \frac{1}{2}\left(\frac{\partial\,\Delta L}{\partial \dot{\vec{\mathbf{r}}}}\right)^2\right] \\ &=\; -\;\left[\,R \;+\; \vec{\mathbf{p}}\cdot(\vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}) \;-\; \vec{\mathbf{a}} \cdot \vec{\mathbf{r}}\,\right] \;=\; -\;\left[\,R \;+\; (\vec{\mathbf{r}} \times \vec{\mathbf{p}})\cdot \vec{\boldsymbol{\mu}} \;-\; \vec{\mathbf{a}} \cdot \vec{\mathbf{r}}\,\right] \;\; , \end{aligned} \tag{48}$$

with vector $\vec{\mathbf{J}} \;=\; \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ being the satellite's orbital angular momentum in the inertial frame.

According to (35) and (47), the momentum can be written as

$$\vec{\mathbf{p}} \;=\; \vec{\mathbf{g}} \;+\; \vec{\boldsymbol{\Phi}} \;+\; \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}} \quad , \tag{49}$$

whence the Hamiltonian perturbation becomes

$$\Delta H \;=\; -\;\left[\,R \;+\; \left(\vec{\boldsymbol{f}} \times \vec{\mathbf{g}}\right)\cdot \vec{\boldsymbol{\mu}} \;+\; \left(\vec{\boldsymbol{\Phi}} \;+\; \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}}\right)\cdot\left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}}\right) - \vec{\mathbf{a}} \cdot \vec{\boldsymbol{f}}\,\right] \;\; . \tag{50}$$

3.3 Elements defined in an accelerated, rotating frame that osculate in the comoving inertial frame

In this subsection we recall a calculation carried out by Goldreich (1965) and Brumberg, Evdokimova & Kochina (1971) and demonstrate that it may be interpreted as an example of nontrivial gauge fixing.

Let us implement the VOP method in a frame that is accelerating at rate $\vec{\mathbf{a}}$ and rotating at angular rate $\vec{\boldsymbol{\mu}}$ relative to some inertial system S. This means that, in the VOP equation (43), ΔL is given by formula (46) and ΔH by (50).

We now choose to describe the motion in the generalised Lagrange gauge (30), so the expression $\left(\vec{\boldsymbol{\Phi}} + \vec{\boldsymbol{\mu}} \times \vec{\mathbf{r}}\right)$ on the right-hand side of (50) vanishes (as follows from (47)), and the expression for ΔH in terms of $\vec{\boldsymbol{f}}$ and $\vec{\mathbf{g}}$ has the form:

$$\Delta H \;=\; -\;\left[\,R(\vec{\boldsymbol{f}},\,t) + \vec{\boldsymbol{\mu}}\cdot(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}}) \;-\; \vec{\mathbf{a}}\cdot\vec{\boldsymbol{f}}\,\right] \quad . \tag{51}$$

At the same time, the generic expression for the VOP given by (25) simplifies to (31). Insertion of (51) therein leads us to

$$[C_r\;C_i]\,\frac{dC_i}{dt} \;=\; \frac{\partial}{\partial C_r}\,\left[\,R(\vec{\boldsymbol{f}},\,t) \;+\; \vec{\boldsymbol{\mu}}\cdot(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}}) \;-\; \vec{\mathbf{a}}\cdot\vec{\boldsymbol{f}}\,\right] \quad . \tag{52}$$

Interestingly, this equation does not contain $\dot{\vec{\boldsymbol{\mu}}}$ even though it is valid for non-uniform precession.

As explained in subsection 2.3, in the generalised Lagrange gauge the vector $\vec{\mathbf{g}}$ is equal to the canonical momentum $\vec{\mathbf{p}} \;=\; \dot{\vec{\mathbf{r}}} \,+\, \partial\,\Delta L/\partial\dot{\vec{\mathbf{r}}}$. In the case when the velocity dependence of ΔL is called into being by inertial forces, the momentum is, according to (47),

$$\vec{\mathbf{p}} \;=\; \dot{\vec{\mathbf{r}}} \,+\, \frac{\partial\,\Delta L}{\partial\dot{\vec{\mathbf{r}}}} \;=\; \dot{\vec{\mathbf{r}}} \,+\, \vec{\boldsymbol{\mu}}\,\times\,\vec{\mathbf{r}} \;\;, \tag{53}$$

which is the particle's velocity relative to the inertial frame comoving with the accelerated, rotating frame. In this sense we may say that our elements are defined in the accelerated, rotating frame but osculate in the comoving inertial one.

In the appendix we provide explicit expression for each of the partial derivatives of $\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{J}}$ that appears in the planetary equations (52).

3.4 Elements defined in the accelerated, rotating frame, that osculate in this frame

Here we not only define the elements in the accelerated, rotating frame, but we also make them osculate in this system, i.e., we make them satisfy $\vec{\boldsymbol{\Phi}} = 0$. In this gauge, expression (50) takes the following form:

$$\Delta H \;=\; -\;\Big[R(\vec{\boldsymbol{f}},\,t) \,+\, \vec{\boldsymbol{\mu}}\cdot(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}}) \;+\; (\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{f}})\cdot(\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{f}}) \;-\; \vec{\mathbf{a}}\cdot\vec{\boldsymbol{f}}\Big] \quad . \tag{54}$$

while equation (25), after some algebra,[5] looks like this:

$$\begin{aligned} [C_n\;C_i]\,\frac{dC_i}{dt} \;&=\; -\,\frac{\partial\,\Delta H}{\partial C_n} \\ &+\vec{\boldsymbol{\mu}}\cdot\left(\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\times\vec{\mathbf{g}} \;-\; \vec{\boldsymbol{f}}\times\frac{\partial\vec{\mathbf{g}}}{\partial C_n}\right) \;-\; \dot{\vec{\boldsymbol{\mu}}}\cdot\left(\vec{\boldsymbol{f}}\times\frac{\partial\vec{\boldsymbol{f}}}{\partial C_n}\right) \;-\; \left(\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{f}}\right)\,\frac{\partial}{\partial C_n}\left(\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{f}}\right) \quad . \end{aligned} \tag{55}$$

[5] Due to (47), the second term on the left-hand side in (25) is proportional to $[\partial(\vec{\boldsymbol{\mu}}\times\vec{\boldsymbol{f}})/\partial C_j]\,\dot{C}_j \;=\; \vec{\boldsymbol{\mu}}\times\dot{\vec{\boldsymbol{\Phi}}}$ and, therefore, vanishes. The second term on the right-hand side simplifies in accordance with the simple rule $\vec{\mathbf{A}}\,\cdot\,(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) \;=\; (\vec{\mathbf{C}}\times\vec{\mathbf{A}})\,\cdot\,\vec{\mathbf{B}}$. (See Efroimsky & (Goldreich 2003b).)

When substituting (54) into (55), it is convenient to rent the expression for ΔH apart and to group the term $(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}}) \cdot (\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}})$ with the last term on the right-hand side of (55):

$$[C_n\; C_i]\,\frac{dC_i}{dt} \;=\; \frac{\partial}{\partial C_n}\left[R(\vec{\boldsymbol{f}},\,t) + \vec{\boldsymbol{\mu}} \cdot (\vec{\boldsymbol{f}} \times \vec{\mathbf{g}}) \;-\; \vec{\mathbf{a}} \cdot \vec{\boldsymbol{f}}\right]$$

$$+\,\vec{\boldsymbol{\mu}} \cdot \left(\frac{\partial \vec{\boldsymbol{f}}}{\partial C_n} \times \vec{\mathbf{g}} \;-\; \vec{\boldsymbol{f}} \times \frac{\partial \vec{\mathbf{g}}}{\partial C_n}\right) \;-\; \dot{\vec{\boldsymbol{\mu}}} \cdot \left(\vec{\boldsymbol{f}} \times \frac{\partial \vec{\boldsymbol{f}}}{\partial C_n}\right) \;+\; \left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}}\right)\,\frac{\partial}{\partial C_n}\left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{f}}\right) \quad . \tag{56}$$

In so writing (56) we have deliberately cast it into a form that eases comparison with (52).

In the appendix we set up an apparatus from which the partial derivatives of the inertial terms with respect to the orbital elements may be obtained. We also show that some of these derivatives vanish. However, a complete evaluation of the inertial input to the planetary equations in the ordinary Lagrange gauge involves a long and tedious calculation which we do not carry out.

3.5 Comparison of the two gauges

One of the powers of gauge freedom lies in the availability of gauge choices that simplify the planetary equations, as we can see from contrasting (52) with (56). While the latter equation is written under the customary Lagrange constraint (i.e., for elements osculating in the frame where they are defined), the former equation is written under a nontrivial constraint called the "generalised Lagrange gauge." The simplicity of (52) speaks for itself.

By identifying the parameters C_i with the Delaunay variables, one arrives from (52) and (56) to the appropriate Delaunay-type equations (see Appendix I to Efroimsky & Goldreich 2003a). The Lagrange equations corresponding to (52) and to (56) may be derived from each of these two equations by choosing C_i as the Kepler elements and using the appropriate Lagrange brackets. These Lagrange equations are written down in the Appendix to Efroimsky & Goldreich (2003b) to which we refer the interested reader.

Although the planetary equations are much simpler in the generalised Lagrange gauge than in the ordinary Lagrange gauge, some of these differences are less important than others. In many physical situations, though not always, the $\vec{\boldsymbol{\mu}}^{\,2}$ and $\dot{\vec{\boldsymbol{\mu}}}$ terms in (56) are of a higher order of smallness compared to those linear in $\vec{\boldsymbol{\mu}}$, and therefore may be neglected, at least for sufficiently short times.[6]

4 Planetary Equations and Gauges in the Hamilton-Jacobi Approach

In this section we demonstrate that the derivation of planetary equations in the N-particle ($N \geq 3$) case, performed through the medium of Hamilton-Jacobi method, implicitly contains a gauge-fixing condition not visible to the naked eye. We present a squeezed account of our study; a comprehensive description containing technical details may be found in Efroimsky & Goldreich (2003a).

The Hamilton-Jacobi analysis rests on the availability of different canonical descriptions of the same physical process. Any two such descriptions, $(\,q,\,p,\,H(q,p)\,)$ and

[6] As an example of an exception to this rule, we mention Venus whose wobble is considerable. This means that, for example, the $\dot{\vec{\boldsymbol{\mu}}}$ term cannot be neglected in computations of circumvenusian orbits.

$(\,Q,\,P,\,H^*(Q,P)\,)$, correspond to different parametrisations of the same phase flow, and both obey Hamilton's equations. Due to the latter circumstance the infinitesimally small variations

$$d\theta \;=\; p\,dq \;-\; H\,dt \tag{57}$$

and

$$d\tilde{\theta} \;=\; P\,dQ \;-\; H^*\,dt \tag{58}$$

are perfect differentials, and so is their difference

$$-\;dW \;\equiv\; d\tilde{\theta} \;-\; d\theta \;=\; P\,dQ \;-\; p\,dq \;-\; (H^* \;-\; H)\;dt \;\;. \tag{59}$$

Here, vectors q, p, Q, and P each contain N components. Given a phase flow parametrised by a set $(\,q,\,p,\,H(q,p,t)\,)$, it is always useful to simplify the description by a canonical transformation to a new set $(\,Q,\,P,\,H^*(Q,P,t)\,)$, with the new Hamiltonian H^* being constant in time. Most advantageous are transformations that nullify the new Hamiltonian H^* , because then the new canonical equations render the variables $(\,Q,\,P\,)$ constant. A powerful method of generating such transformations stems from (59) being a perfect differential. It is sufficient to consider W to be a function of the time and only two other canonical variables, for example q and Q . Then (59) may be written as

$$-\;\frac{\partial W}{\partial t}\;dt\;-\;\frac{\partial W}{\partial Q}\;dQ\;-\;\frac{\partial W}{\partial q}\;dq\;=\;P\,dQ\;-\;p\,dq\;+\;(H\;-\;H^*)\;dt \tag{60}$$

from which it follows that

$$P\;=\;-\;\frac{\partial W}{\partial Q}\quad,\qquad p\;=\;\frac{\partial W}{\partial q}\quad,\qquad H(q,\,p,\,t)\;+\;\frac{\partial W}{\partial t}\;=\;H^*(Q,\,P,\,t)\quad. \tag{61}$$

Inserting the second equation into the third and assuming that $H^*(Q,\,P,\,t)$ is simply a constant, we get the famous Jacobi equation

$$H\left(q,\,\frac{\partial W}{\partial q}\,,\,t\right)\;+\;\frac{\partial W}{\partial t}\;=\;H^* \tag{62}$$

whose solution furnishes the transformation-generating function W. The elegant power of the method becomes most visible if the constant H^* is set to zero. Under this assumption the reduced two-body problem is easily resolved. Starting with the three spherical coordinates and their canonical momenta as $(q,\,p)$, one arrives to canonically conjugate constants $(Q,\,P)$ that coincide with the Delaunay elements (27): $(Q_1\,,\;P_1)\;=\;(L\,,\;-M_o)$; $(Q_2\,,\;P_2)\;=\;(G\,,\;-\omega)$; $(Q_3\,,\;P_3)\;=\;(H\,,\;-\Omega)$.

Extension of this approach to the N-particle problem begins with consideration of a disturbed 2-body setting. The number of degrees of freedom is still the same (three coordinates q and three conjugate momenta p), but the initial Hamiltonian is perturbed:

$$\dot{q}\;=\;\frac{\partial(H\;+\;\Delta H)}{\partial p}\quad,\qquad \dot{p}\;=\;-\;\frac{\partial(H\;+\;\Delta H)}{\partial q}\quad. \tag{63}$$

While in (60) - (62) one begins with the initial Hamiltonian H and ends up with $H^*\;=\;0$, the method may be extended to the perturbed setting by accepting that now we start with a disturbed initial Hamiltonian $H\,+\,\Delta H$ and arrive, through the same canonical transformation, to an equally disturbed eventual Hamiltonian $H^*+\Delta H\;=\;\Delta H$. Plugging

these new Hamiltonians into (60) leads to cancellation of the disturbance ΔH on the right-hand side, whereafter one arrives to the same equation for $W(q,\, Q,\, t)$ as in the unperturbed case. Now, however, the new canonical variables are no longer conserved but obey the dynamical equations

$$\dot{Q} \;=\; \frac{\partial\,\Delta H}{\partial P} \quad , \qquad \dot{P} \;=\; -\;\frac{\partial\,\Delta H}{\partial Q} \quad . \tag{64}$$

Because the same generating function is used in the perturbed and unperturbed cases, the new, perturbed, solution $(q,\, p)$ is expressed through the perturbed "constants" $Q(t)$ and $P(t)$ in the same manner as the old, undisturbed, q and p were expressed through the old constants Q and P. This form-invariance provides the key to the N-particle problem: one should choose the transformation-generating function W to be additive over the particles and repeat this procedure for each of the bodies, separately.

Armed with this preparation, we can proceed to uncover the implicit gauge choice made in using the Hamilton-Jacobi method to derive evolution equations for the orbital elements. To do this we substitute the equalities

$$\dot{Q} \;=\; \frac{\partial\,\Delta H}{\partial P} \;=\; \frac{\partial\,\Delta H}{\partial q}\,\frac{\partial q}{\partial P} \;+\; \frac{\partial\,\Delta H}{\partial p}\,\frac{\partial p}{\partial P} \tag{65}$$

and

$$\dot{P} \;=\; -\;\frac{\partial\,\Delta H}{\partial Q} \;=\; -\;\frac{\partial\,\Delta H}{\partial q}\,\frac{\partial q}{\partial Q} \;-\; \frac{\partial\,\Delta H}{\partial p}\,\frac{\partial p}{\partial P} \tag{66}$$

into the expression for the velocity

$$\dot{q} \;=\; \frac{\partial q}{\partial t} \;+\; \frac{\partial q}{\partial Q}\,\dot{Q} \;+\; \frac{\partial q}{\partial P}\,\dot{P} \quad . \tag{67}$$

This leads to

$$\begin{aligned} \dot{q} \;&=\; \frac{\partial q}{\partial t} \;+\; \left(\frac{\partial q}{\partial Q}\,\frac{\partial q}{\partial P} \;-\; \frac{\partial q}{\partial P}\,\frac{\partial q}{\partial Q}\right)\,\frac{\partial\,\Delta H}{\partial q} \;+\; \left(\frac{\partial q}{\partial Q}\,\frac{\partial p}{\partial P} \;-\; \frac{\partial q}{\partial P}\,\frac{\partial p}{\partial Q}\right)\,\frac{\partial\,\Delta H}{\partial p} \\ &=\; g \;+\; \left(\frac{\partial\,\Delta H}{\partial p}\right)_{q,\,t} \quad , \qquad\qquad g \;\equiv\; \frac{\partial q}{\partial t} \quad , \end{aligned} \tag{68}$$

where we have taken into account that the Jacobian of the canonical transformation is unity:

$$\frac{\partial q}{\partial Q}\,\frac{\partial p}{\partial P} \;-\; \frac{\partial q}{\partial P}\,\frac{\partial p}{\partial Q} \;=\; 1 \quad . \tag{69}$$

To establish the link between the regular VOP method and the canonical treatment, compare (68) with (41). We see that the symplectic description necessarily imposes a particular gauge $\Phi \;=\; \partial\,\Delta H/\partial p$.

It can be easily demonstrated that this special gauge coincides with the generalised Lagrange gauge (30) discussed in subsection 2.2. To that end one has to compare the Hamilton equation for the perturbed Hamiltonian (19),

$$\dot{q} \;=\; \frac{\partial\,\left(H \;+\; \Delta H\right)}{\partial p} \;=\; p \;+\; \frac{\partial\,\Delta H}{\partial p} \quad , \tag{70}$$

with the definition of momentum from the Lagrangian (17),

$$p \equiv \frac{\partial \, (\, L(q, \, \dot{q}, \, t) \, + \, \Delta L(q, \, \dot{q}, \, t) \,)}{\partial \dot{q}} \; = \; \dot{q} \; + \; \frac{\partial \, \Delta L}{\partial \dot{q}} \quad . \tag{71}$$

Equating the above two expressions immediately yields

$$\Phi \; \equiv \; \left(\frac{\partial \, \Delta H}{\partial p} \right)_{q, \, t} \; = \; - \; \left(\frac{\partial \, \Delta L}{\partial \dot{q}} \right)_{q, \, t} \tag{72}$$

which coincides with (30). Thus the transformation generated by $W(q, \, Q, \, t)$ is canonical only if the physical velocity $\dot{q}$ is split in a fashion prescribed by (68), i.e., if (72) is fulfilled. This is exactly what our Theorem from subsection 2.2 says.

To summarise, the generalised Lagrange constraint, $\vec{\mathbf{\Phi}} \; = \; - \; \partial \, \Delta L / \partial \dot{q}$, is tacitly instilled into the Hamilton-Jacobi method. Simply by employing this method (at least, in its straightforward form), we automatically fix the gauge.[7] By sticking to the Hamiltonian description we sacrifice gauge freedom.

Above, in subsection 2.3, we established that in the generalised Lagrange gauge the momentum coincides with $\vec{\mathbf{g}}$. We now can get to the same conclusion from (68), (71) and (72):

$$p \equiv \frac{\partial \, (\, L(q, \, \dot{q}, \, t) \, + \, \Delta L(q, \, \dot{q}, \, t) \,)}{\partial \dot{q}} \; = \; \dot{q} \; - \; \Phi \; = \; g \quad . \tag{73}$$

Thus, implementation of the Hamilton-Jacobi theory in celestial mechanics demands the orbital elements to osculate in phase space. Naturally, this demand reduces to that of regular osculation in the simple case of velocity-independent ΔL .

5 Conclusions

In the article thus far we have studied the topic recently raised in the literature: the planetary equations' internal symmetry that stems from the freedom of supplementary condition's choice. The necessity of making such a choice constrains the trajectory to a 9-dimensional submanifold of the 12-dimensional space spanned by the orbital elements and their time derivatives. Similarly to the field theory, the choice of the constraint (= the choice of gauge) is vastly ambiguous and reveals a hidden symmetry instilled in the description of the N-body problem in the language of orbital elements.

We addressed the issue of internal freedom in a sufficiently general setting where a perturbation to the two-body problem depends not only upon positions but also upon velocities. Such situations emerge when relativistic corrections to Newton's law are taken into account or when the VOP method is employed in rotating systems of reference.

Just as a choice of an appropriate gauge simplifies solution of the equations of motion in electrodynamics, an alternative (to that of Lagrange) choice of gauge in the celestial mechanics can simplify orbit calculations. We provided one such example,a satellite orbiting a precessing planet. In this example, the choice of the generalised Lagrange gauge considerably simplifies matters.

[7] An explanation of this phenomenon from a different viewpoint is offered in Section 6 of Efroimsky (2003) where the Delaunay equations are derived also through a direct change of variables. It turns out that the outcome retains the symplectic form only if an extra constraint is imposed by hand.

We have explained where the Lagrange constraint tacitly enters the Hamilton-Jacobi derivation of the Delaunay equations. This constraint turns out to be an inseparable (though not easily visible) part of the method: in the case of momentum-independent disturbances, the N-body generalisation of the 2-body Hamilton-Jacobi technique is legitimate only if we use orbital elements that are osculating. In the situation where the disturbance depends not only upon positions but also upon velocities, another constraint (which we call the "generalised Lagrange constraint") turns out to be stiffly embedded into the Hamilton-Jacobi development of the problem.

Unless a specific constraint (gauge) is imposed by hand, the planetary equations assume their general, gauge-invariant, form. In the case of a velocity-independent disturbance, any gauge different from that of Lagrange drives the Delaunay system away from its symplectic form. If we permit the disturbing force to depend also upon velocities, the Delaunay equations retain their canonicity only in the generalised Lagrange gauge. Interestingly, in this special gauge the instantaneous ellipses (hyperbolae) osculate in phase space.

Briefly speaking, N-body celestial mechanics, expressed in terms of orbital elements, is a gauge theory but it is not strictly canonical. It becomes canonical in the generalised Lagrange gauge.

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Appendix

In this appendix we set up an apparatus from which one may evaluate the partial derivatives with respect to the orbital elements of inertial terms that appear in the planetary equations derived in sections 3.3 and 3.4. We then show that some of these derivatives vanish. Following that, we derive explicit expressions for each derivative of $\vec{\boldsymbol{\mu}} \cdot \left(\vec{\boldsymbol{f}} \times \vec{\mathbf{g}}\right)$, which provides a complete analytic evaluation of the rotational input in the generalised Lagrange gauge. The topic is further developed (and the appropriate generalised Lagrange system of equations is presented) in (Efroimsky & Goldreich 2003b).

To find the explicit form of the dependence $\vec{\boldsymbol{f}} = \vec{\boldsymbol{f}}(C_i,\, t)$, it is conventional to introduce an auxiliary set of Cartesian coordinates $\vec{\mathbf{q}}$, with an origin at the gravitating centre, and with the first two axes located in the plane of orbit. The $\vec{\mathbf{q}}$ coordinates are easy to express through the major semiaxis a , the eccentricity e and the eccentric anomaly E :

$$q_1 \;\equiv\; a\,(\cos E \;-\; e) \quad , \qquad q_2 \;\equiv\; a\sqrt{1\,-\,e^2}\;\sin E \quad , \qquad q_3 \;=\; 0 \quad , \tag{74}$$

where E itself is a function of the major semiaxis a , the eccentricity e , the mean anomaly at epoch, M_0 , and the time, t . The time dependence is realised through the Kepler equation

$$E \,-\, e\,\sin E \;=\; M \quad , \tag{75}$$

where

$$M \;\equiv\; M_o \,+\, \mu^{1/2}\,\int_{t_o}^{t}\; a^{-3/2}\; dt \quad . \tag{76}$$

The inertial-frame-related position of the body reads:

$$\vec{\boldsymbol{r}} \;=\; \vec{\boldsymbol{f}}\,(\Omega,\, i,\, \omega,\, a,\, e,\, M_o\,;\; t) \;=\; \hat{\mathbf{R}}(\Omega,\, i,\, \omega)\;\vec{\mathbf{q}}\,(a,\, e,\; E(a,\, e,\, M_o,\, t)\,) \quad , \tag{77}$$

$\hat{\mathbf{R}}(\Omega,\, i,\, \omega)$ being the matrix of rotation from the orbital-plane-related coordinate system $\mathbf{q}$ to the fiducial frame $(x,\, y,\, z)$ in which the vector $\vec{\mathbf{r}}$ is defined. This rotation is parametrised by the three Euler angles: inclination, i ; the longitude of the node, Ω ; and the argument of the pericentre, ω .

In the unperturbed two-body setting the velocity is expressed by

$$\vec{\mathbf{g}} \;=\; \frac{\partial}{\partial t}\,\vec{\boldsymbol{f}}\,(\Omega,\, i,\, \omega,\, a,\, e,\, M_o\,;\; t) \;=\; \left(\frac{\partial E}{\partial t}\right)_{a,\,e,\,M_o} \hat{\mathbf{R}}(\Omega,\, i,\, \omega)\;\left(\frac{\partial \vec{\mathbf{q}}}{\partial E}\right)_{a,\,e} \quad . \tag{78}$$

One can similarly calculate partial derivatives of $\vec{\boldsymbol{f}}$ with respect to M_o :

$$\frac{\partial}{\partial M_o}\,\vec{\boldsymbol{f}}\,(\Omega,\, i,\, \omega,\, a,\, e,\, M_o\,;\; t) \;=\; \left(\frac{\partial E}{\partial M_o}\right)_{a,\,e,\,t} \hat{\mathbf{R}}(\Omega,\, i,\, \omega)\;\left(\frac{\partial \vec{\mathbf{q}}}{\partial E}\right)_{a,\,e} \quad , \tag{79}$$

whence it becomes evident that $\partial\vec{\boldsymbol{f}}/\partial M_o$ is parallel to $\vec{\mathbf{g}}$ and, hence,

$$\vec{\mathbf{g}}\;\times\;\left(\frac{\partial \vec{\boldsymbol{f}}}{\partial M_o}\right)_{\Omega,\,i,\,\omega,\,a,\,e,\,t} \;=\; 0 \quad . \tag{80}$$

By a similar trick it is possible to demonstrate that $\partial(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}})/\partial M_o$ is proportional to $\partial(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}})/\partial E$ and, therefore, to $\partial(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}})/\partial t$. Hence, this derivative vanishes (because in the two-particle case the cross product $\vec{\boldsymbol{f}}\times\vec{\mathbf{g}}$ is an integral of motion). This vanishing of $\partial(\vec{\boldsymbol{f}}\times\vec{\mathbf{g}})/\partial M_o$, along with (80), implies:

$$\vec{\boldsymbol{f}}\;\times\;\left(\frac{\partial \vec{\mathbf{g}}}{\partial M_o}\right)_{\Omega,\,i,\,\omega,\,a,\,e,\,t} \;=\; 0 \quad . \tag{81}$$

In the situation when the parameters C_i are implemented by the Delaunay elements, a similar sequence of calculations leads to

$$\vec{\mathbf{g}}\;\times\;\left(\frac{\partial \vec{\boldsymbol{f}}}{\partial M_o}\right)_{\Omega,\,\omega,\,L,\,G,\,H,\,t} \;=\; 0 \tag{82}$$

and, appropriately, to

$$\vec{\boldsymbol{f}}\;\times\;\left(\frac{\partial \vec{\mathbf{g}}}{\partial M_o}\right)_{\Omega,\,\omega,\,L,\,G,\,H,\,t} \;=\; 0 \quad . \tag{83}$$

We can proceed much farther in the generalised Lagrange gauge, at least in so far as derivatives of the rotational input $\vec{\mathbf{J}}/\mu \;=\; \vec{\boldsymbol{f}}\times\vec{\boldsymbol{g}}$ are concerned. (We remind that here and everywhere μ stands for the reduced mass, while $\vec{\boldsymbol{\mu}}$ denotes the precession rate.)

As we proved above, this cross product is independent of M_o and, hence,

$$\vec{\boldsymbol{\mu}}\;\cdot\;\frac{\partial(\vec{\boldsymbol{f}}\times\vec{\boldsymbol{g}})}{\partial M_o} = 0 \quad . \tag{84}$$

Since $\vec{\mathbf{J}}$ is orthogonal to the orbit plane, it is invariant under rotations of the orbit within its plane, whence

$$\vec{\boldsymbol{\mu}} \cdot \frac{\partial(\vec{\boldsymbol{f}} \times \vec{\boldsymbol{g}})}{\partial \omega} = 0 \ . \tag{85}$$

To continue, we note that in the two-body setting the ratio $\vec{\mathbf{J}}/\mu$, is known to be equal to $\sqrt{a(1 - e^2)}\ \hat{\mathbf{w}}$ where $\hat{\mathbf{w}}$ is a unit vector perpendicular to the unperturbed orbit's plane. Moreover, in planet-associated noninertial axes $(x,\, y,\, z)$ with corresponding unit vectors $(\hat{\mathbf{x}},\, \hat{\mathbf{y}},\, \hat{\mathbf{z}})$, the normal to the orbit is expressed by

$$\hat{\mathbf{w}} \ = \ \hat{\mathbf{x}}\ \sin i\ \sin\Omega \ - \ \hat{\mathbf{y}}\ \sin i\ \cos\Omega \ + \ \hat{\mathbf{z}}\ \cos i \ . \tag{86}$$

Hence,

$$\vec{\boldsymbol{\mu}} \cdot \frac{\partial\left(\vec{\boldsymbol{f}} \times \vec{\boldsymbol{g}}\right)}{\partial a} \ = \ \vec{\boldsymbol{\mu}} \cdot \hat{\mathbf{w}}\ \frac{\partial\left(\sqrt{a(1 - e^2)}\right)}{\partial a} \ = \ \frac{1}{2}\sqrt{\frac{1 - e^2}{a}}\ \mu_{\perp} \ , \tag{87}$$

and

$$\vec{\boldsymbol{\mu}} \cdot \frac{\partial\left(\vec{\boldsymbol{f}} \times \vec{\boldsymbol{g}}\right)}{\partial e} \ = \ \vec{\boldsymbol{\mu}} \cdot \hat{\mathbf{w}}\ \frac{\partial\left(\sqrt{a(1 - e^2)}\right)}{\partial e} \ = \ -\sqrt{\frac{a\ e^2}{1 - e^2}}\ \mu_{\perp} \ , \tag{88}$$

where $\mu_{\perp} \ = \ \mu_x\ \sin i\ \sin\Omega \ - \ \mu_y\ \sin i\ \cos\Omega \ + \ \mu_z\ \cos i$ is the orthogonal-to-orbit component of the precession rate. The remaining two derivatives look:

$$\vec{\boldsymbol{\mu}} \cdot \frac{\partial\left(\vec{\boldsymbol{f}} \times \vec{\boldsymbol{g}}\right)}{\partial \Omega} \ = \ \sqrt{a\ (1 - e^2)}\ \ \vec{\boldsymbol{\mu}} \cdot \frac{\partial \hat{\mathbf{w}}}{\partial \Omega} \ = \\ \sqrt{a\ (1 - e^2)}\ \ \{\mu_x\ \sin i\ \cos\Omega \ + \ \mu_y\ \sin i\ \sin\Omega\} \tag{89}$$

and

$$\vec{\boldsymbol{\mu}} \cdot \frac{\partial\left(\vec{\boldsymbol{f}} \times \vec{\boldsymbol{g}}\right)}{\partial i} \ = \ \sqrt{a\ (1 - e^2)}\ \ \vec{\boldsymbol{\mu}} \cdot \frac{\partial \hat{\mathbf{w}}}{\partial i} \ = \\ \sqrt{a\ (1 - e^2)}\ \ \{\mu_x\ \cos i\ \sin\Omega \ - \ \mu_y\ \cos i\ \cos\Omega \ - \ \mu_z\ \sin i\} \tag{90}$$

As for the derivatives of $\vec{\mathbf{a}} \cdot \vec{\boldsymbol{f}}$, they may be calculated directly from the expression for $\vec{\boldsymbol{f}}(\Omega,\, \omega,\, i,\, a,\, e,\, M_o\, ;\ t)$ presented above. However, the resulting expressions are cumbersome so we do not present them here.

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